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LEARNING RESOURCES

RULES OF DIFFERENTIATION

Differentiation is all about measuring change!
Measuring change in a linear function:

$$y = a + bx$$

a = intercept

b = constant slope i.e. the impact of a unit change in x on the level of y

$$\mathbf{b} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of the graph of a function is called the derivative of the function

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

- The process of differentiation involves letting the change in x become arbitrarily small, i.e. letting $\Delta x \to 0$
- e.g if = $2X+\Delta X$ and $\Delta X \to 0$
- $\Rightarrow$ = 2X in the limit as $\Delta X \to 0$

Rules for Differentiation
(section 4.3)

1. The Constant Rule

If $y = c$ where c is a constant,

$$\frac{dy}{dx} = 0$$

e.g. $y = 10$ then $\frac{dy}{dx} = 0$

2. The Power Function Rule

If $y = ax^n$, where a and n are constants

$$\frac{dy}{dx} = n.a.x^{n-1}$$

i) $y = 4x \Rightarrow \frac{dy}{dx} = 4x^0 = 4$

ii) $y = 4x^2 \Rightarrow \frac{dy}{dx} = 8x$

iii) $y = 4x^{-2} \Rightarrow \frac{dy}{dx} = -8x^{-3}$

3. The Sum-Difference Rule

If $y = f(x) \pm g(x)$

$$\frac{dy}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$$

If y is the sum/difference of two or more functions of x:

differentiate the 2 (or more) terms separately, then add/subtract

(i) $y = 2x^2 + 3x$ then $\frac{dy}{dx} = 4x + 3$

(ii) $y = 5x + 4$ then $\frac{dy}{dx} = 5$

4. The Product Rule

$y = u.v.w$

If $y = u.v$ where u and v are functions of x, ($u = f(x)$ *and* $v = g(x)$) Then

$$\frac{dy}{dx} = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

$$\frac{dy}{dx} = uv\,\frac{dw}{dx} + uw\,\frac{dv}{dx} + vw\,\frac{du}{dx}$$

Examples

If $y = u.v$ $\qquad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

i) $y = (x+2)(ax^2+bx)$

$$\frac{dy}{dx} = (x+2)(2ax+b) + (ax^2+bx)$$

ii) $y = (4x^3-3x+2)(2x^2+4x)$

$$\frac{dy}{dx} = (4x^3-3x+2)(4x+4) + (2x^2+4x)(12x^2-3)$$

5. The Quotient Rule

- If y = *u/v* where *u* and *v* are functions of x (*u = f(x) and v = g(x)*) Then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$If \quad y = \frac{u}{v} \qquad then \qquad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1

$$y = \frac{(x+2)}{(x+4)}$$

$$\frac{dy}{dx} = \frac{(x+4)(1) - (x+2)(1)}{(x+4)^2} = \frac{-2}{(x+4)^2}$$

6. The Chain Rule
(Implicit Function Rule)

- If y is a function of v, and v is a function of x, then y is a function of x and

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

y = f(x)

y = f I where I = f(x)

Examples

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$$

i) $y = (ax^2 + bx)^{½}$

let $v = (ax^2 + bx)$, so $y = v^{½}$

$$\frac{dy}{dx} = \frac{1}{2}\left(ax^2 + bx\right)^{-\frac{1}{2}} .(2ax + b)$$

ii) $y = (4x^3 + 3x - 7)^4$

let $v = (4x^3 + 3x - 7)$, so $y = v^4$

$$\frac{dy}{dx} = 4\left(4x^3 + 3x - 7\right)^3 .\left(12x^2 + 3\right)$$

7. Differentiation of Power on expressions in Brackets

$$y = (5x^2 + 6x)^{3/2}$$

Let $y = (u)^{3/2}$

$$u = 5x^2 + 6x$$

by appling chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}(u)^{3/2} \cdot \frac{d}{dx}(5x^2 + 6x)$$

$$= \frac{3}{2} u^{\frac{3}{2} - 1} \cdot 10x + 6$$

$$= \frac{3}{2} u^{\frac{1}{2}} \cdot 10x + 6$$

Sub. $u = 5x^2 + 6x$

sub $u = 5x^2 + 6x$

$$= \frac{3}{2}(u)^{1/2} \cdot 10x+6$$

$$= \frac{3}{2}\left(5x^2+6x\right)^{\frac{1}{2}} \cdot 10x+6$$

$$= \frac{3}{2}\sqrt{5x^2+6x} \cdot 10x+6$$

// Ans

8.Differentiation of Root of Expression

If $Y = \sqrt{5x^3 - 3}$ find $\frac{dy}{dx}$.

by appling chain rule

$$y = \sqrt{u} = u^{1/2}$$

$$u = 5x^3 - 3$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{d}{dx}(u^{1/2}) \cdot \frac{d}{dx}(5x^3 - 3)$$

$$= \frac{1}{2} u^{\frac{1}{2}-1} \cdot 15x^2$$

$$= \frac{1}{2} u^{-\frac{1}{2}} \cdot 15x^2$$

Sub. $u = 5x^3 - 3$

$$\frac{1}{2}(5x^3 - 3)^{-\frac{1}{2}} \cdot 15x^2$$

$$= \frac{15x^2}{2}(5x^3 - 3)^{-\frac{1}{2}}$$

$$= \frac{15x^2}{2} \cdot \frac{1}{(5x^3 - 3)^{1/2}}$$

$$= \frac{15x^2}{2} \cdot \frac{1}{\sqrt{5x^3 - 3}}$$

8. The Inverse Function Rule

If x = f(y) then $\frac{dy}{dx} = \frac{1}{dx/dy}$

- Examples

i) $x = 3y^2$ then

$$\frac{dx}{dy} = 6y \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{6y}$$

ii) $y = 4x^3$ then

$$\frac{dy}{dx} = 12x^2 \quad \text{so} \quad \frac{dx}{dy} = \frac{1}{12x^2}$$

Differentiation of Implicit Function

① $x^2y + y - 2x = 0$ Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 y) + \frac{d}{dx}(y) - \frac{d}{dx}(2x) = \frac{d}{dx}(0)$$

consider x^2y as uv, so applying uv method

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$u = x^2 \quad v = y$$

$$\frac{dy}{dx} = y\frac{d}{dx}(x^2) + x^2\frac{d}{dx}(y)$$

$$= y2x + x^2\frac{dy}{dx}$$

$$= y2x + x^2\frac{dy}{dx} + \frac{dy}{dx} - 2 = 0$$

$$x^2\frac{dy}{dx} + \frac{dy}{dx} = -2xy + 2$$

$$x^2 \frac{dy}{dx} + \frac{dy}{dx} = -2xy + 2$$

Taking $\frac{dy}{dx}$ as common

$$\frac{dy}{dx}(x^2 + 1) = -2xy + 2$$

$$\frac{dy}{dx} = \frac{-2xy + 2}{x^2 + 1}$$

// Ans

(i) $x^2 - y^2 + 3x = 5y$ Find $\frac{dy}{dx}$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(y^2) + \frac{d}{dx}(3x) = \frac{d}{dx}(5y)$$

$$2x - 2y\frac{dy}{dx} + 3 = 5\frac{dy}{dx}$$

$$-2y\frac{dy}{dx} - 5\frac{dy}{dx} = -2x - 3$$

eliminating (-)

$$2y\frac{dy}{dx} + 5\frac{dy}{dx} = 2x + 3$$

Taking $\frac{dy}{dx}$ as common

$$\frac{dy}{dx}(2y+5) = 2x+3$$

$$\frac{dy}{dx} = \frac{2x+3}{2y+5}$$ // Ans.

Differentiation of logarithmic Function

1) $y = \log x$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \frac{1}{x} \quad //$$

2) $y = \log 7x$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}(\log 7x)$$

$$= \frac{1}{7x} \cdot 7 = \frac{7}{7x}$$

$$= \frac{1}{x} \quad //$$

3) $y = \log(x^2 + 2x)$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}(\log x^2 + 2x)$$

$$= \frac{1}{x^2 + 2x}(2x + 2)$$

$$= \frac{2x + 2}{x^2 + 2x} \quad //$$

$y = \log 5x +$

find $\frac{dy}{dx}$.

7. Differentiation of Power on expressions in Brackets

$$y = (5x^2 + 6x)^{3/2}$$

Let $y = (u)^{3/2}$

$$u = 5x^2 + 6x$$

by applying chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{d}{du}(u)^{3/2} \cdot \frac{d}{dx}(5x^2 + 6x$$

$$= \frac{3}{2} u^{\frac{3}{2} - 1} \cdot 10x + 6$$

$$= \frac{3}{2} u^{\frac{1}{2}} \cdot 10x + 6$$

sub. $u = 5x^2 + 6x$

sub $u = 5x^2 + 6x$

$$= \frac{3}{2}(u)^{1/2} \cdot 10x + 6$$

$$= \frac{3}{2}(5x^2 + 6x)^{\frac{1}{2}} \cdot 10x + 6$$

$$= \frac{3}{2}\sqrt{5x^2 + 6x} \cdot 10x + 6$$ // Ans

Differentiation in Economics

Application I

- Total Costs = TC = FC + VC
- Total Revenue = TR = P * Q
- π = Profit = TR – TC
- Break even: π = 0, or TR = TC
- Profit Maximisation: MR = MC

Application I: Marginal Functions (Revenue, Costs and Profit)

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Calculating Marginal Functions

$$MR = \frac{d(TR)}{dQ}$$

$$MC = \frac{d(TC)}{dQ}$$

Example 1

- A firm faces the demand curve P=17-3Q
- (i) Find an expression for TR in terms of Q
- (ii) Find an expression for MR in terms of Q

Solution:

$TR = P.Q = 17Q - 3Q^2$

$$MR = \frac{d(TR)}{dQ} = 17 - 6Q$$

Example 2

A firms total cost curve is given by

$$TC=Q3- 4Q^2+12Q$$

(i) Find an expression for AC in terms of Q

(ii) Find an expression for MC in terms of Q

(iii) When does AC=MC?

(iv) When does the slope of AC=0?

(v) Plot MC and AC curves and comment on the economic significance of their relationship

Solution

(i) $\mathbf{TC = Q^3 - 4Q^2 + 12Q}$

Then, **AC** $= {}^{TC}/_{Q} = Q^2 - 4Q + 12$

(ii) MC $= \dfrac{d(TC)}{dQ} = 3Q^2 - 8Q + 12$

(iii) When does AC = MC?

$Q^2 - 4Q + 12 = 3Q^2 - 8Q + 12$

$\Rightarrow Q = 2$

Thus, AC = MC when Q = 2

Solution continued....

(iv) When does the slope of AC = 0?

$$\frac{d(AC)}{dQ} = 2Q - 4 = 0$$

$\Rightarrow$ Q = 2 when slope AC = 0

(v) Economic Significance?

MC cuts AC curve at minimum point..

9. Differentiating Exponential Functions

If $y = \exp(x) = e^x$ where $e = 2.71828\ldots$

$$\text{then } \frac{dy}{dx} = e^x$$

More generally,

If $y = Ae^{rx}$

$$\text{then } \frac{dy}{dx} = rAe^{rx} = ry$$

Examples

1) $y = e^{2x}$ then $\frac{dy}{dx} = 2e^{2x}$

2) $y = e^{-7x}$ then $\frac{dy}{dx} = -7e^{-7x}$

10. Differentiating Natural Logs

Recall if $\mathbf{y = e^x}$ then $\mathbf{x = \log_e y = \ln y}$

- If $y = e^x$ then $\frac{dy}{dx} = e^x = y$

- **From The Inverse Function Rule**

$$y = e^x \Rightarrow \frac{dx}{dy} = \frac{1}{y}$$

- Now, if $\mathbf{y = e^x}$ this is equivalent to writing
  $\mathbf{x = \ln y}$

- Thus, $\mathbf{x = \ln y} \Rightarrow \frac{dx}{dy} = \frac{1}{y}$

More generally,

if $\mathbf{y} = \ln \mathbf{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

NOTE: the derivative of a natural log function does not depend on the co-efficient of x

Thus, if $\mathbf{y} = \ln \mathbf{mx} \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

Proof

- if y = ln mx   m>0
- **Rules of Logs** $\Rightarrow$ y = ln m+ ln x
- **Differentiating** (Sum-Difference rule)

$$\frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$

Examples

1) $y = \ln 5x \quad (x>0) \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

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2) $y = \ln(x^2+2x+1)$

let $v = (x^2+2x+1)$ so $y = \ln v$

Chain Rule: $\Rightarrow \frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{1}{x^2 + 2x + 1} \cdot (2x + 2)$$

$$\frac{dy}{dx} = \frac{(2x + 2)}{(x^2 + 2x + 1)}$$

3) $y = x^4 \ln x$

Product Rule: ⇒

$$\frac{dy}{dx} = x^4 \frac{1}{x} + \ln x . 4x^3$$

$$= x^3 + 4x^3 \ln x \quad = x^3 (1 + 4 \ln x)$$

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4) $y = \ln(x^3(x+2)^4)$

Simplify first using rules of logs

$\Rightarrow y = \ln x^3 + \ln(x+2)^4$

$\Rightarrow y = 3\ln x + 4\ln(x+2)$

$$\frac{dy}{dx} = \frac{3}{x} + \frac{4}{x+2}$$

Applications II

- how does demand change with a change in price……
- e_d=

$$\frac{proportional\ change\ in\ demand}{proportional\ change\ in\ price}$$

$$= \frac{\Delta Q}{Q} \Big/ \frac{\Delta P}{P} = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q}$$

Point elasticity of demand

$$e_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

e_d is negative for a downward sloping demand curve

–Inelastic demand if $| e_d | < 1$

–Unit elastic demand if $| e_d | = 1$

–Elastic demand if $| e_d | > 1$

Example 1

Find e_d of the function $Q = aP^{-b}$

$$e_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

$$e_d = -baP^{-b-1} \cdot \frac{P}{aP^{-b}}$$

$$= \frac{-baP^{-b}}{P} \cdot \frac{P}{aP^{-b}} = -b$$

e_d at all price levels is $-b$

Example 2

If the (inverse) Demand equation is

P = 200 – 40ln(Q+1)

Calculate the price elasticity of demand when Q = 20

- Price elasticity of demand: $e_d = \frac{dQ}{dP} \cdot \frac{P}{Q} < 0$
- P is expressed in terms of Q,

$$\frac{dP}{dQ} = -\frac{40}{Q+1}$$

- Inverse rule $\Rightarrow \frac{dQ}{dP} = -\frac{Q+1}{40}$
- Hence, $e_d = -\frac{Q+1}{40} \cdot \frac{P}{Q} < 0$
- Q is 20 $\Rightarrow e_d = -\frac{21}{40} \cdot \frac{78.22}{20} = -2.05$

(where P = 200 – 40ln(20+1) = 78.22)

Application III: Differentiation of Natural Logs to find *Proportional* Changes

The derivative of log(f(x)) ≡ $f'(x)/_{f(x)}$, or the proportional change in the variable x

i.e. y = f(x), then the proportional Δ x

$$= \frac{dy}{dx} \cdot \frac{1}{y} = \frac{d(\ln y)}{dx}$$

Take logs and differentiate to find proportional changes in variables

1) Show that if $y = x^{\alpha}$, then $\frac{dy}{dx} \cdot \frac{1}{y} = \frac{\alpha}{x}$

and this ≡ derivative of ln(y) with respect to x.

Solution:

$$\frac{dy}{dx} \cdot \frac{1}{y} = \frac{1}{y} . \alpha \, x^{\alpha - 1}$$

$$= \frac{1}{y} . \alpha \; \frac{x^{\alpha}}{x}$$

$$= \frac{1}{y} . \alpha \, . \frac{y}{x}$$

$$= \frac{\alpha}{x}$$

Solution Continued...

Now $\ln y = \ln x^{\alpha}$

Re-writing $\Rightarrow \ln y = \alpha \ln x$

$$\Rightarrow \frac{d(\ln\ y)}{dx} = \alpha \cdot \frac{1}{x} = \frac{\alpha}{x}$$

Differentiating the ln y with respect to x gives the proportional change in x.

Example 2: If Price level at time t is

$P(t) = a+bt+ct^2$

Calculate the rate of inflation.

Solution:

The inflation rate at t is the proportional change in p

$$\frac{1}{P(t)} \cdot \frac{dP(t)}{dt} = \frac{b+2ct}{a+bt+ct^2}$$

Alternatively,

differentiating the log of P(t) wrt t directly

$\ln P(t) = \ln(a+bt+ct^2)$

where $v = (a+bt+ct^2)$ so $\ln P = \ln v$

Using chain rule,

$$\frac{d(\ln P(t))}{dt} = \frac{b+2ct}{a+bt+ct^2}$$

Higher Order Differentiation

$\frac{dy}{dx} = \underline{I}^{st}$ order differentiation

$\frac{d^2y}{dx^2} = \underline{II}^{st}$,, ,,

$\frac{d^3y}{dx^3} = \underline{III}^{rd}$,, ,,

Maxima and Minima

Condition For Maxima
Or
Condition for Maxima Value

1) $\frac{dy}{dx} = f'x = 0$

2) $\frac{d^2y}{dx^2} = f''x < 0$

Condition For Minima
Or
Condition for Minima Value

1) $\frac{dy}{dx} = f'x = 0$

2) $\frac{d^2y}{dx^2} = f''x > 0$

Find maxima and minima

$$y = x^3 - 3x^2 + 20$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 3x^2 + 20)$$

$$= 3x^2 - 6x$$

x either 0 (or) 2

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(3x^2 - 6x)$$

$$= 6x - 6$$

If $x = 2$

$6x - 6$

$6(2) - 6$

$12 - 6 = 6$

$6 > 0$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 3 \quad b = -6 \quad c = 0$

$$\frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot 0}}{2 \cdot 3}$$

$$= \frac{6 \pm \sqrt{36 - 0}}{6}$$

$$= \frac{6 \pm 6}{6} = \frac{6+6}{6} = \frac{12}{6} = 2$$

$$= \frac{6-6}{6} = \frac{0}{6} = 0$$

sub $x = 0$

$6x - 6$

$6(0) - 6$

$0 - 6$

$-6 < 0$

The function is maximum when $x = 0$

The function is minimum when $x = 2$

Find the Maximum and minimum values of Function $2X^3+3X^2-12X-6$

Let $y = 2x^3 + 3x^2 - 12x - 6$

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^3 + 3x^2 - 12x - 6\right)$$

$$= 6x^2 + 6x - 12$$

$$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}\left(6x^2 + 6x - 12\right)$$

$$= 12x + 6$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 6 \quad b = +6$

$c = -12$

$$\frac{-6 \pm \sqrt{6^2 - 4 \cdot 6 \cdot -12}}{2 \cdot 6}$$

$$= \frac{-6 \pm \sqrt{36 + 288}}{12}$$

Sub. $x = 1$

$12x + 6$

$12(1) + 6$

$12 + 6$

$= 18 > 0$

The function is minimum when $x = 1$

Sub $x = -2$

$12x + 6$

$12(-2) + 6$

$-24 + 6$

$-18 < 0$

The function is maximum when $x = -2$

$$\frac{-6 \pm \sqrt{324}}{12}$$

$$\frac{-6 \pm 18}{12} = \frac{-6 + 18}{12}$$

$$= \frac{12}{12} = 1 \quad x = 1$$

$$= \frac{-6 - 18}{12}$$

$$= \frac{-24}{12} = -2$$

$$x = -2$$

$x = 1$

$x = -2$

The Minimum Value of the Function

$2x^3 + 3x^2 - 12x - 6$

$x = 1$

$2(1)^3 + 3(1)^2 - 12(1) - 6 = 2 + 3 - 12 - 6 = -13$

The Minimum Value of the Function = -13

The Maximum Value of the Function $2X^3+3X^2-12X-6$

$x = -2$

$= 2(-2)^3 + 3(-2)^2 - 12(-2) - 6 = 2(-8) + 3(4) + 24 - 6$

$= -16 + 12 + 24 - 6 = 14$

The Maximum Value of the Function = 14